Electromagnetic scattering from a cavity embedded in an impedance ground plane

Kui Du

School of Mathematical Sciences
Xiamen University

Joint work with
Weiwei Sun and Buyang Li

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Outline

(1) Introduction to the model

(2) Well-posedness of the 2D problem

(3) Numerical results

(4) Future work

The electromagnetic cavity problem

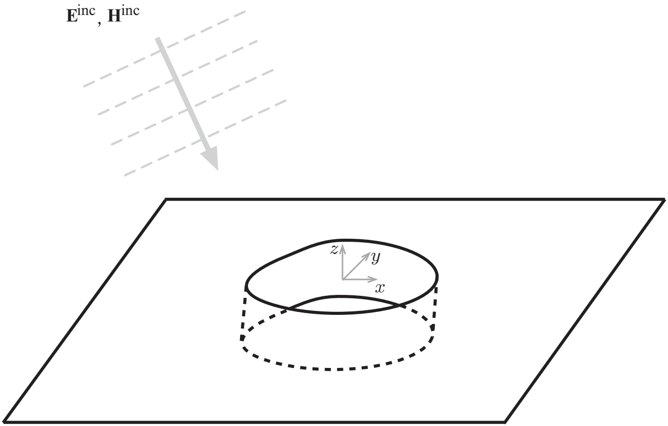


Figure: Cavity geometry

Governing equations

- Total fields in the upper-half space and the cavity:

$$\mathbf{E} = \mathbf{E}^{\text{inc}} + \mathbf{E}^{\text{r}} + \mathbf{E}^{\text{s}},$$

$$\mathbf{H} = \mathbf{H}^{\text{inc}} + \mathbf{H}^{\text{r}} + \mathbf{H}^{\text{s}}.$$

- Time-harmonic Maxwell's equations (time dependence $e^{-\mathrm{i}\omega t}$):

$$\nabla \times \mathbf{E} - \mathrm{i}\omega\mu\mathbf{H} = 0,$$

$$\nabla \times \mathbf{H} + \mathrm{i}\omega\varepsilon\mathbf{E} = 0,$$

  where $\mathrm{i} = \sqrt{-1}$ is the imaginary unit, ω is the angular frequency and the physical parameters ε and μ denote, respectively, the permittivity (farads/meter) and the permeability (henrys/meter) of the medium.

Boundary conditions and Radiation conditions

- At a perfectly conducting surface:

$$\mathbf{n} \times \mathbf{E} = 0, \qquad \mathbf{n} \cdot \mathbf{H} = 0.$$

- At an imperfectly conducting surface:

$$\frac{1}{\mu_r^+}\mathbf{n} \times (\nabla \times \mathbf{E}) - \frac{\mathrm{i}k_0}{\eta}\mathbf{n} \times (\mathbf{n} \times \mathbf{E}) = 0,$$

$$\frac{1}{\varepsilon_r^+}\mathbf{n} \times (\nabla \times \mathbf{H}) - \mathrm{i}k_0\eta\mathbf{n} \times (\mathbf{n} \times \mathbf{H}) = 0,$$

  $\mathbf{n}$: the unit normal pointing into the ground,
  $k_0 = \omega\sqrt{\varepsilon_0\mu_0} > 0$: the free space wave number,
  $\eta = \sqrt{\mu_r^- / \varepsilon_r^-}$: the normalized intrinsic impedance.

- Radiation conditions at infinity:

$$\lim_{r\to\infty} r \left[\nabla \times \begin{pmatrix} \mathbf{E}^{\mathrm{s}} \\ \mathbf{H}^{\mathrm{s}} \end{pmatrix} - \mathrm{i}k_0\hat{r} \times \begin{pmatrix} \mathbf{E}^{\mathrm{s}} \\ \mathbf{H}^{\mathrm{s}} \end{pmatrix}\right] = 0.$$

The 2D cavity problem

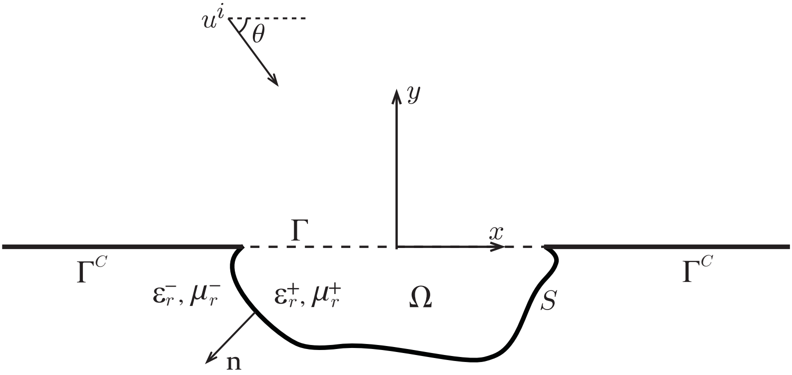


Figure: Cross section of the geometry

The 2D model: E_z polarization

The electric field is parallel to the z-axis.

- $\mathbf{E} = (0, 0, E_z)$
- $\mathbf{H} = (H_x, H_y, 0)$
- $\mathbf{n} = (n_x, n_y, 0)$

The problem is governed by

$$\begin{cases} \nabla \cdot \left(\dfrac{1}{\mu_r^+} \nabla E_z \right) + k_0^2 \varepsilon_r^+ E_z = 0, & \text{in} \quad \mathbb{R}_+^2 \cup \Omega, \\ \dfrac{1}{\mu_r^+} \dfrac{\partial E_z}{\partial n} - \dfrac{\mathrm{i} k_0}{\eta} E_z = 0, & \text{on} \quad \Gamma^C \cup S, \\ \lim\limits_{r \to \infty} \sqrt{r} \left(\dfrac{\partial E_z^{\mathrm{s}}}{\partial r} - \mathrm{i} k_0 E_z^{\mathrm{s}} \right) = 0 & \text{at} \quad \text{infinity}. \end{cases}$$

The 2D model: H_z polarization

The magnetic field is parallel to the z-axis.

- $\mathbf{E} = (E_x, E_y, 0)$
- $\mathbf{H} = (0, 0, H_z)$
- $\mathbf{n} = (n_x, n_y, 0)$

The problem is governed by

$$\begin{cases} \nabla \cdot \left(\dfrac{1}{\varepsilon_r^+} \nabla H_z \right) + k_0^2 \mu_r^+ H_z = 0, & \text{in} \quad \mathbb{R}_+^2 \cup \Omega, \\ \dfrac{1}{\varepsilon_r^+} \dfrac{\partial H_z}{\partial n} - \mathrm{i} k_0 \eta H_z = 0, & \text{on} \quad \Gamma^C \cup S, \\ \lim\limits_{r \to \infty} \sqrt{r} \left(\dfrac{\partial H_z^{\mathrm{s}}}{\partial r} - \mathrm{i} k_0 H_z^{\mathrm{s}} \right) = 0 & \text{at} \quad \text{infinity.} \end{cases}$$

The 2D model: unified form

- The E_z and H_z polarizations can be written in the unified form

$$\begin{cases} \nabla \cdot \left(\frac{1}{a} \nabla u \right) + k_0^2 b u = 0, & \text{in} \quad \mathbb{R}^2_+ \cup \Omega, \\ \frac{1}{a} \frac{\partial u}{\partial n} - \mathrm{i}\rho u = 0, & \text{on} \quad \Gamma^C \cup S, \\ \lim_{r \to \infty} \sqrt{r} \left(\frac{\partial u^{\mathrm{s}}}{\partial r} - \mathrm{i} k_0 u^{\mathrm{s}} \right) = 0 & \text{at} \quad \text{infinity}. \end{cases}$$

  - Total field $u(x, y) = E_z$ or H_z, $u = u^i + u^{\mathrm{r}} + u^{\mathrm{s}}$
  - $a(x, y)$, $b(x, y)$: complex scalar functions of position
    $a(x, y) = 1$ and $b(x, y) = 1$ in $\mathbb{R}^2_+$,
    $\mathrm{Re}(a) \geq a_0 > 0$, $\mathrm{Im}(a) \geq 0$, $\mathrm{Re}(b) \geq b_0 > 0$, and $\mathrm{Im}(b) \geq 0$,
  - $\rho \in \mathbb{C}$ is a constant with $\mathrm{Re}(\rho) > 0$ or $\rho = 0$.

The 2D scattering problem

The 2D scattering problem reads: for a given incident plane wave u^i, determine the scattered field u^{s} in the cavity and the upper half-plane.

- The incident field u^i is given by

$$u^i = e^{\mathrm{i}k_0(x\cos\theta - y\sin\theta)},$$

  where $0 < \theta < \pi$ is the angle of incidence with respect to the positive x-axis.

- The total field

$$u = u^i + u^{\mathrm{r}} + u^{\mathrm{s}}$$

  where u^{r} is the reflected field due to the infinite impedance ground plane, u^{s} is the unknown scattered field

The reflected and scattered fields

- Note that $u^i + u^{\mathrm{r}}$ satisfies

$$\begin{cases} \Delta(u^i + u^{\mathrm{r}}) + k_0^2(u^i + u^{\mathrm{r}}) = 0, & \text{in} \quad \mathbb{R}^2_+, \\ \dfrac{\partial(u^i + u^{\mathrm{r}})}{\partial n} - \mathrm{i}\rho(u^i + u^{\mathrm{r}}) = 0, & \text{on} \quad \{y = 0\}. \end{cases}$$

  The reflected field u^{r} by the infinite impedance ground plane,

$$u^{\mathrm{r}} = -\frac{\rho - k_0 \sin\theta}{\rho + k_0 \sin\theta} e^{\mathrm{i}k_0(x\cos\theta + y\sin\theta)}.$$

- The scattered field u^{s} satisfies

$$\begin{cases} \Delta u^{\mathrm{s}} + k_0^2 u^{\mathrm{s}} = 0, & \text{in} \quad \mathbb{R}^2_+, \\ \dfrac{\partial u^{\mathrm{s}}}{\partial n} - \mathrm{i}\rho u^{\mathrm{s}} = 0, & \text{on} \quad \Gamma^C, \\ u^{\mathrm{s}} = u - g = u - (u^i + u^{\mathrm{r}}), & \text{on} \quad \Gamma. \end{cases}$$

Half-plane impedance Green's function

Let $\mathbf{x} = (x, y) \in \mathbb{R}^2_+$ be the fixed source point, $\mathbf{x_0} = (x_0, y_0)$

The impedance Green's function $G_\rho(\mathbf{x}, \mathbf{x_0})$ satisfies

$$\begin{cases} \Delta_{\mathbf{x_0}} G_\rho(\mathbf{x}, \mathbf{x_0}) + k_0^2 G_\rho(\mathbf{x}, \mathbf{x_0}) = -\delta(\mathbf{x} - \mathbf{x_0}), & \text{in} \quad \mathbb{R}^2_+, \\ \dfrac{\partial G_\rho(\mathbf{x}, \mathbf{x_0})}{\partial n(\mathbf{x_0})} - \mathrm{i}\rho G_\rho(\mathbf{x}, \mathbf{x_0}) = 0, & \text{on} \quad \{y_0 = 0\}. \end{cases}$$

We have

$$\begin{aligned} G_\rho(\mathbf{x}, \mathbf{x_0}) &= \frac{1}{4\pi} \int_{-\infty}^{\infty} e^{-\sqrt{\xi^2 - k_0^2}|y_0 - y|} \frac{e^{\mathrm{i}(x_0 - x)\xi}}{\sqrt{\xi^2 - k_0^2}} \mathrm{d}\xi \\ &- \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{\mathrm{i}\rho + \sqrt{\xi^2 - k_0^2}}{\mathrm{i}\rho - \sqrt{\xi^2 - k_0^2}} e^{-\sqrt{\xi^2 - k_0^2}(y_0 + y)} \frac{e^{\mathrm{i}(x_0 - x)\xi}}{\sqrt{\xi^2 - k_0^2}} \mathrm{d}\xi. \end{aligned}$$

The complex square root is characterized, for $\xi, k_0 \in \mathbb{R}$, by

$$\sqrt{\xi^2 - k_0^2} = \begin{cases} \sqrt{\xi^2 - k_0^2}, & \text{if} \quad |\xi| \geq k_0, \\ -\mathrm{i}\sqrt{k_0^2 - \xi^2}, & \text{if} \quad |\xi| < k_0. \end{cases}$$

Half-plane Dirichlet/Neumann Green's function

We have the following remark for the impedance Green's function:

- Dirichlet boundary condition: $\rho = \infty$

$$G_\infty(\mathbf{x}, \mathbf{x_0}) = \frac{\mathrm{i}}{4}\left[H_0^{(1)}(k_0|\mathbf{x}-\mathbf{x_0}|) - H_0^{(1)}(k_0|\mathbf{x}-\bar{\mathbf{x}}_\mathbf{0}|)\right].$$

- Neumann boundary condition: $\rho = 0$

$$G_0(\mathbf{x}, \mathbf{x_0}) = \frac{\mathrm{i}}{4}\left[H_0^{(1)}(k_0|\mathbf{x}-\mathbf{x_0}|) + H_0^{(1)}(k_0|\mathbf{x}-\bar{\mathbf{x}}_\mathbf{0}|)\right].$$

Interior problem: Dirichlet case $\rho = \infty$

By the Green's function method, we have

$$\begin{cases} \nabla \cdot \left(\dfrac{1}{a} \nabla u \right) + k_0^2 b u = 0, & \text{in} \quad \Omega, \\ u = 0, & \text{on} \quad S, \\ \dfrac{\partial u}{\partial n} = \mathcal{T}(u) + g, & \text{on} \quad \Gamma, \end{cases}$$

where

$$\mathcal{T}(u) := \frac{\mathrm{i}k_0}{2} \fint_{\Gamma} \frac{1}{|x-t|} H_1^{(1)}(k_0|x-t|) u(t,0) \mathrm{d}t.$$

Interior problem: Dirichlet case $\rho = \infty$

Let

$$U := \{w \in H^1(\Omega) : w = 0 \text{ on } S \text{ and } w|_\Gamma \in H_{00}^{1/2}(\Gamma)\}.$$

The variational formulation: Find $u \in U$ such that

$$A(u, w) = G(w), \qquad \forall w \in U$$

where

$$A(u, w) = \int_\Omega \left(\frac{1}{a} \nabla u \cdot \overline{\nabla w} - k_0^2 b u \overline{w} \right) \mathrm{d}x \mathrm{d}y - \int_\Gamma \mathcal{T}(u) \overline{w} \mathrm{d}x$$

and

$$G(w) = \int_G g \overline{w} \mathrm{d}x.$$

The variational problem has a unique solution $u \in U$.

Interior problem: $\rho \neq \infty$

By the Green's function method, we have

$$\begin{cases} \nabla \cdot \left(\dfrac{1}{a} \nabla u \right) + k_0^2 b u = 0, & \text{in} \quad \Omega, \\ \dfrac{1}{a} \dfrac{\partial u}{\partial n} - \mathrm{i}\rho u = 0, & \text{on} \quad S, \\ u(\mathbf{x}) = g(\mathbf{x}) \\ \qquad - \displaystyle\int_\Gamma G_\rho(\mathbf{x}, \mathbf{x_0}) \left(\dfrac{1}{a} \dfrac{\partial u}{\partial y}(\mathbf{x_0}) + \mathrm{i}\rho u(\mathbf{x_0}) \right) \mathrm{d}s(\mathbf{x_0}), & \text{on} \quad \Gamma, \end{cases}$$

where

$$\begin{aligned} G_\rho(\mathbf{x}, \mathbf{x_0}) &= -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{\mathrm{i}(x_0 - x)\xi}}{\mathrm{i}\rho - \sqrt{\xi^2 - k_0^2}} \mathrm{d}\xi \\ &= -\frac{1}{\pi} \int_0^{\infty} \frac{\cos\left((x_0 - x)\xi\right)}{\mathrm{i}\rho - \sqrt{\xi^2 - k_0^2}} \mathrm{d}\xi, \quad \forall\, \mathbf{x}, \mathbf{x_0} \in \Gamma. \end{aligned}$$

Interior problem: $\rho \neq \infty$

Find $(u, w) \in H^1(\Omega) \times \widetilde{H}^{-1/2}(\Gamma)$ such that

$$B(u, w; v, \varphi) = l(v, \varphi), \quad \forall\, (v, \varphi) \in H^1(\Omega) \times \widetilde{H}^{-1/2}(\Gamma),$$

where $l(v, \varphi) = \int_\Gamma g\varphi \mathrm{d}x$ and

$$B(u, w; v, \varphi) = \int_\Omega \left(\frac{1}{a} \nabla u \cdot \nabla v - k_0^2 buv \right) \mathrm{d}x\mathrm{d}y - \int_S \rho uv \mathrm{d}s \\ - \int_\Gamma wv \mathrm{d}x + \int_\Gamma \mathbf{G}_\rho(w + \rho u)\, \varphi \mathrm{d}x + \int_\Gamma u\varphi \mathrm{d}x,$$

where the operator $\mathbf{G}_\rho : \widetilde{H}^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ is defined by

$$\mathbf{G}_\rho w(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} m(\xi) \widehat{\widetilde{w}}(\xi) e^{\mathrm{i}x\xi} \mathrm{d}\xi,$$

where

$$m(\xi) := \frac{1}{\sqrt{\xi^2 - k_0^2 - \mathrm{i}\rho}}, \quad \widehat{\widetilde{w}}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widetilde{w}(x) e^{-\mathrm{i}x\xi} \mathrm{d}x.$$

Interior problem: $\rho \neq \infty$

Theorem 1 (Existence and uniqueness)

Suppose that $k_0 > 0$, ρ is a complex constant with either

$$\mathrm{Re}(\rho) > 0 \quad \text{or} \quad \rho = 0,$$

and $a, b \in L^\infty(\Omega)$ are complex scalar functions such that

$$\mathrm{Re}(a) \geq a_0 > 0, \quad \mathrm{Im}(a) \geq 0, \quad \mathrm{Re}(b) \geq b_0 > 0, \quad \mathrm{Im}(b) \geq 0.$$

Then, there exists a unique solution

$$(u, w) \in H^1(\Omega) \times \widetilde{H}^{-1/2}(\Gamma)$$

for the variational problem.

Radar cross section

The physical parameter of interest is the RCS defined by

$$\sigma(\vartheta) \quad = \quad \lim_{r\to\infty} 2\pi r \frac{|u^{\mathrm{s}}(r\cos\vartheta, r\sin\vartheta)|^2}{|u^i|^2}$$

where ϑ is the observation angle with respect to the positive x-axis. When the incident and observation directions are the same ($\theta = \vartheta$), we have the backscatter RCS

$$\text{Backscatter RCS}(\vartheta) = 10\log_{10}\sigma(\vartheta)\ \text{dB}.$$

By the impedance boundary condition, the field continuity conditions, and the far field behavior of the impedance Green' function G_ρ, we can evaluate $\sigma(\vartheta)$ as

$$\sigma(\vartheta) = \frac{4}{k_0}|P(\vartheta)|^2,$$

where $P(\vartheta)$ is the far-field coefficient given by

$$P(\vartheta) = \frac{1}{2}\frac{k_0\sin\vartheta}{\rho + k_0\sin\vartheta}\int_\Gamma \left(\frac{1}{a}\frac{\partial u}{\partial y} + \mathrm{i}\rho u\right)\mathrm{e}^{\mathrm{i}k_0 x\cos\vartheta}\mathrm{d}x.$$

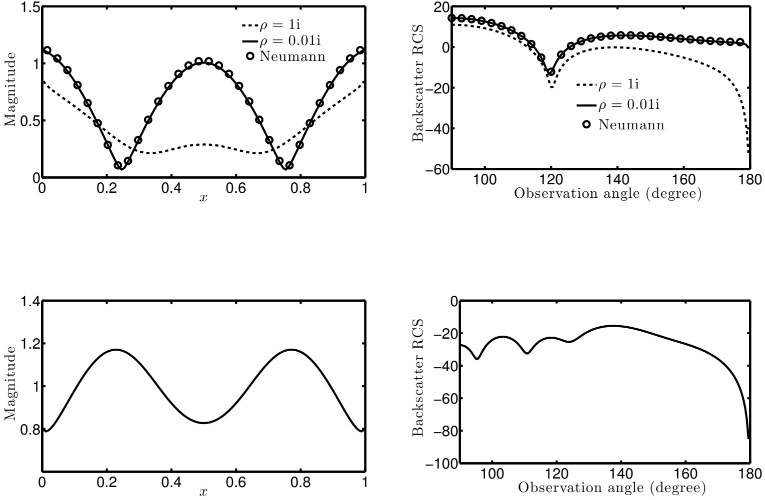
Numerical simulation

We report computational results for a rectangular cavity with 1 meter wide and 0.25 meter deep ($L = 1.0$ and $D = 0.25$). Our focus is on the efficiency of the proposed model and the finite difference method for RCS calculation. Two different cases (see Figure 3) are considered.

Figure: The empty cavity (left) and the filled cavity (right).

The empty cavity

$a(x,y) = b(x,y) = 1$ in Ω.



The filled cavity

$$a(x,y) = \begin{cases} 4+\mathrm{i}, & 0.2 < x < 0.8, \quad -0.25 < y < -0.20, \\ 1, & \text{otherwise}, \end{cases}$$

$$b(x,y) = 1.$$

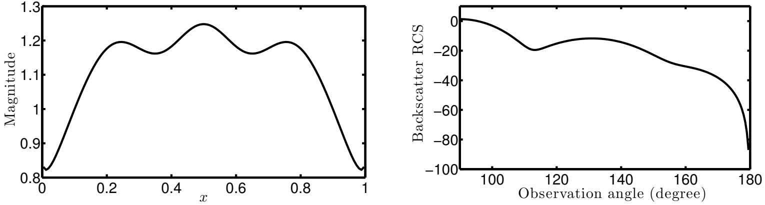
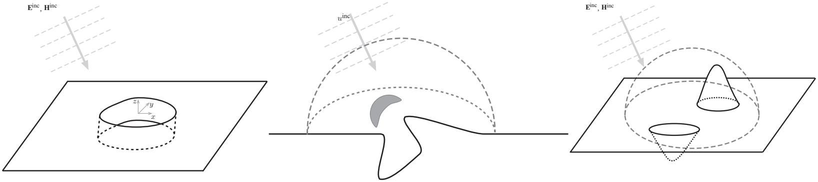


Figure: Aperture field (left) at normal incidence $\theta = \pi/2$ and backscatter RCS (right) for the filled cavity with $k_0 = 4\pi$ and $\rho = k_0$.

Future work



- Well-posedness
- High accuracy methods for the (hyper)singular integrals
- The adaptive method
- Fast algorithm/Preconditioning + Deflation

Thank you!