Proof of Theorem 1. <induction on j>.

We prove if $\vec{r}_j \neq 0$, $j = 0, 1, 2, \cdots, l$ then $\forall\ 1 \le j \le l$

<1> $\vec{r}_i^{\,*} \vec{r}_j = 0$ and $\vec{p}_i^{\,*} A \vec{p}_j = 0 \quad \forall\ i < j$

<2> $\vec{p}_j \neq 0$

<3> $\operatorname{span}\{\vec{r}_0, A\vec{r}_0, \cdots, A^j \vec{r}_0\} = \operatorname{span}\{\vec{x}_1 - \vec{x}_0, \vec{x}_2 - \vec{x}_0, \cdots, \vec{x}_{j+1} - \vec{x}_0\}$

$= \operatorname{span}\{\vec{p}_0, \vec{p}_1, \cdots, \vec{p}_j\} = \operatorname{span}\{\vec{r}_0, \vec{r}_1, \cdots, \vec{r}_j\}$

$\boxed{j = 1:}$ $\alpha_1 = \dfrac{\vec{r}_0^{\,*} \vec{r}_0}{\vec{p}_0^{\,*} A \vec{p}_0} > 0$: $\quad \vec{r}_0^{\,*} \vec{r}_1 = \vec{r}_0^{\,*} \vec{r}_0 - \alpha_1 \vec{r}_0^{\,*} A \vec{r}_0 = 0.$

$\beta_1 = \dfrac{\vec{r}_1^{\,*} \vec{r}_1}{\vec{r}_0^{\,*} \vec{r}_0} > 0$: $\quad \vec{p}_0^{\,*} A \vec{p}_1 = \vec{p}_0^{\,*} A (\vec{r}_1 + \beta_1 \vec{p}_0) = \vec{p}_0^{\,*} A \vec{r}_1 + \beta_1 \vec{p}_0^{\,*} A \vec{p}_0 \qquad A\vec{p}_0 = \dfrac{\vec{r}_0 - \vec{r}_1}{\alpha_1}$

$= \dfrac{(\vec{r}_0^{\,*} - \vec{r}_1^{\,*}) \vec{r}_1}{\alpha_1} + \dfrac{\beta_1}{\alpha_1} \vec{r}_0^{\,*} \vec{r}_0 = 0.$

$\because \vec{r}_1 \perp \vec{p}_0 \quad \therefore \vec{p}_1 = \vec{r}_1 + \beta_1 \vec{p}_0 \neq \vec{0}.$

$\operatorname{span}\{\vec{x}_1 - \vec{x}_0, \vec{x}_2 - \vec{x}_0\} = \operatorname{span}\{\vec{p}_0, \vec{p}_1\} = \operatorname{span}\{\vec{r}_0, \vec{r}_1\} = \operatorname{span}\{\vec{r}_0, A\vec{r}_0\}$

$\alpha_1 > 0$; $\vec{x}_1 - \vec{x}_0 = \alpha_1 \vec{p}_0$

$\alpha_2 = \dfrac{\vec{r}_1^{\,*} \vec{r}_1}{\vec{p}_1^{\,*} A \vec{p}_1} > 0$

$\vec{x}_2 - \vec{x}_0 = \vec{x}_1 - \vec{x}_0 + \alpha_2 \vec{p}_1$

$\vec{p}_1 = \frac{1}{\alpha_2} (\vec{x}_2 - \vec{x}_0 - (\vec{x}_1 - \vec{x}_0))$

$\vec{p}_0 = \vec{r}_0$
$\vec{p}_1 = \vec{r}_1 + \beta_1 \vec{p}_0$
$\vec{r}_1 = \vec{p}_1 - \beta_1 \vec{p}_0$

$\vec{r}_1 = \vec{r}_0 - \alpha_1 A \vec{p}_0 \Rightarrow$ "$\subseteq$"
$\dim \operatorname{span}\{\vec{r}_0, \vec{r}_1\} = 2.$
$\Rightarrow$ "$=$"

$\boxed{\forall\ j \le l-1}$ $\quad \vec{r}_i^{\,*} \vec{r}_j = 0$ and $\vec{p}_i^{\,*} A \vec{p}_j = 0 \quad \forall\ i < j$

induction hypothesis

$\vec{p}_j \neq \vec{0}$

$\operatorname{span}\{\vec{r}_0, A\vec{r}_0, \cdots, A^j \vec{r}_0\} = \operatorname{span}\{\vec{x}_1 - \vec{x}_0, \vec{x}_2 - \vec{x}_0, \cdots, \vec{x}_{j+1} - \vec{x}_0\}$

$= \operatorname{span}\{\vec{p}_0, \vec{p}_1, \cdots, \vec{p}_j\} = \operatorname{span}\{\vec{r}_0, \vec{r}_1, \cdots, \vec{r}_j\}$

$\boxed{j = l.}$ $\quad \forall\ i < l$ i.e., ($i < l-1$ and $i = l-1$).

$\vec{r}_i^{\,*} \vec{r}_l = \vec{r}_i^{\,*} (\vec{r}_{l-1} - \alpha_l A \vec{p}_{l-1}) = \underbrace{\vec{r}_i^{\,*} \vec{r}_{l-1}}_{=0} - \alpha_l \underbrace{\vec{r}_i^{\,*} A \vec{p}_{l-1}}_{=0} \quad \begin{cases} \vec{r}_i \in \operatorname{span}\{\vec{p}_0, \vec{p}_1, \cdots, \vec{p}_i\} \\ \text{By hypothesis.} \end{cases}$

$\alpha_l = \dfrac{\vec{r}_{l-1}^{\,*} \vec{r}_{l-1}}{\vec{p}_{l-1}^{\,*} A \vec{p}_{l-1}} > 0.$ $\quad$ if $i < l-1$

if $i = l-1$.

$$\vec{r}_i^* \vec{r}_l = \vec{r}_{l-1}^* \vec{r}_{l-1} - \frac{\vec{r}_{l-1}^* \vec{r}_{l-1}}{\vec{p}_{l-1}^* A \vec{p}_{l-1}} \cdot \vec{r}_{l-1}^* A \vec{p}_{l-1} \qquad \langle \vec{p}_{l-1} = \vec{r}_{l-1} + \beta_{l-1} \vec{p}_{l-2} \rangle$$

$$= \vec{r}_{l-1}^* \vec{r}_{l-1} - \vec{r}_{l-1}^* \vec{r}_{l-1} \cdot \frac{\vec{r}_{l-1}^* A \vec{p}_{l-1}}{(r_{l-1} + \beta_{l-1} p_{l-2})^* A p_{l-1}} = 0.$$

$$\vec{p}_i^* A \vec{p}_l = \underbrace{\vec{p}_i^* A \vec{r}_l}_{=0} + \beta \underbrace{\vec{p}_i^* A \vec{p}_{l-1}}_{=0}$$

By hypothesis

if $i < l-1$:

$$\langle A\vec{p}_i = \frac{\vec{r}_i - \vec{r}_{i+1}}{\alpha_{i+1}} \rangle \rightarrow = \frac{\vec{r}_i^* \vec{r}_l - \vec{r}_{i+1}^* \vec{r}_l}{\alpha_{i+1}} = 0.$$

if $i = l-1$:

$$\vec{p}_i^* A \vec{p}_l = \vec{p}_{l-1}^* A \vec{r}_l + \beta_l \vec{p}_{l-1}^* A \vec{p}_{l-1}$$

$$= \frac{(\vec{r}_{l-1}^* - \vec{r}_l^*) \vec{r}_l}{\alpha_l} + \beta_l \vec{p}_{l-1}^* A \vec{p}_{l-1}$$

$$= -\frac{\vec{r}_l^* \vec{r}_l}{\alpha_l} + \frac{\beta_l}{\alpha_l} \cdot \vec{r}_{l-1}^* \vec{r}_{l-1} = 0. \qquad \langle \beta_l = \frac{\vec{r}_l^* \vec{r}_l}{\vec{r}_{l-1}^* \vec{r}_{l-1}} \rangle.$$

$\because \vec{p}_{l-1} \perp \vec{r}_l \qquad \therefore \vec{p}_l = \vec{r}_l + \beta_l \vec{p}_{l-1} \neq \vec{0}$

$$\text{span}\{\vec{x}_1 - \vec{x}_0, \vec{x}_2 - \vec{x}_0, \cdots, \vec{x}_{l+1} - \vec{x}_0\} = \text{span}\{\vec{p}_0, \vec{p}_1, \cdots, \vec{p}_l\}$$

$$\alpha_{l+1} = \frac{\vec{r}_l^* \vec{r}_l}{\vec{p}_l^* A \vec{p}_l} > 0.$$

$$\vec{x}_{l+1} - \vec{x}_0 = \vec{x}_l - \vec{x}_0 + \alpha_{l+1} \vec{p}_j$$

$$\vec{p}_l = \frac{1}{\alpha_{l+1}} (\vec{x}_{l+1} - \vec{x}_0 - (\vec{x}_l - \vec{x}_0))$$

$$= \text{span}\{\vec{r}_0, \vec{r}_1, \cdots, \vec{r}_l\}$$

$$\vec{p}_l = \vec{r}_l + \beta_l \vec{p}_{l-1}$$

$$\vec{r}_l = \vec{p}_l - \beta_l \vec{p}_{l-1}$$

$$= \text{span}\{\vec{r}_0, A\vec{r}_0, \cdots, A^l \vec{r}_0\}$$

$$\vec{r}_l = \vec{r}_{l-1} - \alpha_l A \vec{p}_{l-1} \Rightarrow \text{“}\subseteq\text{”}$$

$$\dim \text{span}\{\vec{r}_0, \vec{r}_1, \cdots, \vec{r}_l\} = l+1 \Rightarrow \text{“}=\text{”}.$$

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