Numerical Linear Algebra Assignment 20

Exercise 1. (10 points)

For a lower-triangular system $\mathbf{Lx} = \mathbf{b}$, we have the following forward elimination algorithm:

$$\begin{cases} x_1 = b_1/l_{11} \\ x_2 = (b_2 - x_1 l_{21})/l_{22} \\ \quad \vdots \\ x_m = (b_m - \sum_{i=1}^{m-1} x_i l_{mi})/l_{mm} \end{cases}$$

This algorithm is backward stable in the sense that the computed solution $\widetilde{\mathbf{x}} \in \mathbb{C}^m$ satisfies

$$(\mathbf{L} + \delta\mathbf{L})\widetilde{\mathbf{x}} = \mathbf{b}$$

for some lower-triangular matrix $\delta\mathbf{L} \in \mathbb{C}^{m\times m}$ with

$$\|\delta\mathbf{L}\|/\|\mathbf{L}\| = \mathcal{O}(\epsilon_{\text{machine}}).$$

Specifically, for each i, j,

$$\frac{|\delta l_{ij}|}{|l_{ij}|} \leq m\epsilon_{\text{machine}} + \mathcal{O}(\epsilon_{\text{machine}}^2).$$

Prove the case $m = 3$.

Exercise 2. (10 points)

Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$. Prove that the algorithm computing the inner product problem

$$f(\mathbf{x}, \mathbf{y}) = \mathbf{x}^\top \mathbf{y} = \sum_{i=1}^{m} x_i y_i$$

by $\otimes$ and $\oplus$ is backward stable.

Exercise 3. (TreBau Exercise 16.1, 10 points)

(a) Let orthogonal matrices $\mathbf{Q}_1, \cdots, \mathbf{Q}_k \in \mathbb{R}^{m\times m}$ be fixed and consider the problem of computing, for $\mathbf{A} \in \mathbb{R}^{m\times n}$, the product $\mathbf{B} = \mathbf{Q}_k \cdots \mathbf{Q}_1 \mathbf{A}$. Let the computation be carried out from right to left by straightforward floating point operations on a computer satisfying the desired properties. Show that this algorithm is backward stable. (Here $\mathbf{A}$ is thought of as data that can be perturbed; the matrices $\mathbf{Q}_j$ are fixed and not to be perturbed.)

(b) Give an example to show that this result no longer holds if the orthogonal matrices $\mathbf{Q}_j$ are replaced by arbitrary matrices $\mathbf{X}_j \in \mathbb{R}^{m\times m}$.