Numerical Linear Algebra Assignment 10

Exercise 1. (Zhihao Cao, 10 points)

Let

$$\mathbf{A} = \begin{bmatrix} a_1 & b_1 & & & \\ c_1 & a_2 & b_2 & & \\ & c_2 & a_3 & \ddots & \\ & & \ddots & \ddots & b_{m-1} \\ & & & c_{m-1} & a_m \end{bmatrix} \in \mathbb{R}^{m\times m}, \qquad c_j b_j > 0.$$

Prove that there exists a diagonal matrix $\mathbf{D}$ satisfying that $\mathbf{D}^{-1}\mathbf{A}\mathbf{D}$ is symmetric.

Exercise 2. (10 points)

How many eigenvalues does $\mathbf{A} = \begin{bmatrix} -2 & 2 & & \\ 2 & 2 & 1 & \\ & 1 & 2 & -1 \\ & & -1 & 1 \end{bmatrix}$ have in the interval $[1, 2]$? Determine the answer via Sturm sequences.

Exercise 3. (10 points)

Prove Proposition 12 of Lecture 10.

Exercise 4. (TreBau Exercise 30.3, 10 points)

Show that if the largest (in magnitude) off-diagonal entry is annihilated at each step of the Jacobi algorithm, then the sum of the squares of the off-diagonal entries decreases by at least the factor $1 - 2/(m^2 - m)$ at each step.

Exercise 5. (TreBau Exercise 31.3, 10 points)

Show that if the entries on both principal diagonals of a bidiagonal matrix are all nonzero, then the singular values of the matrix are distinct.

Exercise 6. (Programming, TreBau Exercise 30.5, 10 points)

Write a program to find the eigenvalues of an $m \times m$ real symmetric matrix by the Jacobi algorithm with the standard row-wise ordering, plotting the sum of the squares of the off-diagonal entries on a log scale as a function of the number of sweeps. Apply your program to random matrices of dimensions 20, 40, and 80.