# Numerical Linear Algebra Assignment 9

### Exercise 1. (10 points)

Consider the simultaneous iteration:

**Algorithm 1**: Simultaneous iteration

Pick $\mathbf{Q}_n^{(0)} \in \mathbb{C}^{m\times n}$ with orthonormal columns
**for** $k = 1, 2, 3, \ldots,$
  $\mathbf{Q}_n^{(k)}\mathbf{R}_n^{(k)} = \mathbf{A}\mathbf{Q}_n^{(k-1)}$  (QR factorization)
**end**

Assume $\mathbf{A} = \mathbf{S}\mathbf{\Lambda}\mathbf{S}^{-1}$ is diagonalizable with $\mathbf{\Lambda} = \mathrm{diag}\{\lambda_1, \lambda_2, \cdots, \lambda_m\}$ and

$$|\lambda_1| \geq \cdots \geq |\lambda_n| > |\lambda_{n+1}| \geq \cdots \geq |\lambda_m|.$$

Assume $\mathbf{X}_n := \begin{bmatrix}\mathbf{I}_n & \mathbf{0}\end{bmatrix}\mathbf{S}^{-1}\mathbf{Q}_n^{(0)}$ has full rank. Prove that

$$\mathbf{Q}_n^{(k)} = \mathbf{A}^k\mathbf{Q}_n^{(0)}(\mathbf{R}_n^{(1)})^{-1}(\mathbf{R}_n^{(2)})^{-1}\cdots(\mathbf{R}_n^{(k)})^{-1}.$$

### Exercise 2. (Zhihao Cao, 10 points)

Let $\mathbf{A}_1 = \begin{bmatrix} a & b \\ \varepsilon & c\end{bmatrix} \in \mathbb{R}^{2\times 2}$. Let $\mathbf{A}_2 = \mathbf{R}_1\mathbf{Q}_1 + c\mathbf{I} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{bmatrix}$, where $\mathbf{Q}_1\mathbf{R}_1 = \mathbf{A}_1 - c\mathbf{I}$ is a QR factorization of $\mathbf{A}_1 - c\mathbf{I}$. Prove:

(a) if $a$ and $c$ are not close (i.e., $\exists\delta > 0$, $|a - c| > \delta$), then $a_{21} = \mathcal{O}(\varepsilon^2)$;

(b) if further $b = \varepsilon$ (i.e., $\mathbf{A}_1$ is symmetric), then $a_{21} = \mathcal{O}(\varepsilon^3)$.

### Exercise 3. (Programming, 10 points)

Write a function `[Q,H] = myhess(A)` that reduces a complex square matrix to upper Hessenberg form by unitary similarity transformations, i.e., $\mathbf{H} = \mathbf{Q}^*\mathbf{A}\mathbf{Q}$. Your program should use only elementary Matlab operations – not the function `hess`, for example. Apply your program to `A = randn(5) + 1i*randn(5)`.

### Exercise 4. (Programming, TreBau Exercise 29.1, 10 points)