Numerical Linear Algebra Assignment 8

Exercise 1. (TreBau Exercise 25.2, 10 points)

Let $e_1, e_2, e_3, \cdots$ be a sequence of nonnegative numbers representing errors in some iterative process that converge to zero, and suppose there are a constant C and an exponent α such that for all sufficiently large k, $e_{k+1} \le C(e_k)^\alpha$. Then, (1) *linear convergence* or *geometric convergence*: $\alpha = 1$ and $C < 1$; (2) *quadratic convergence*: $\alpha = 2$; (3) *cubic convergence*: $\alpha = 3$.

(a) Suppose we want an answer of accuracy $\mathcal{O}(\varepsilon_{\text{machine}})$. Assuming the amount of work for each step is $\mathcal{O}(1)$, show that the total work requirement in the case of linear convergence is $\mathcal{O}(|\log(\varepsilon_{\text{machine}})|)$. How does the constant C enter into your work estimate?

(b) Show that in the case of superlinear convergence, i.e., $\alpha > 1$, the work requirement becomes $\mathcal{O}(\log(|\log(\varepsilon_{\text{machine}})|))$. (Hint: The problem may be simplified by defining a new error measure $f_k = C^{1/(\alpha-1)} e_k$.) How does the exponent α enter into your work estimate?

Exercise 2. (Theorem 1 of Lecture 8, 10 points)

Let $\{\lambda, \mathbf{v}\}$ be an eigenpair of the matrix $\mathbf{A}$, i.e., $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$. We further assume that $\|\mathbf{v}\|_2 = 1$. Prove the following:

(a) If $\mathbf{A}$ is non-normal, then the Rayleigh quotient $r(\mathbf{x})$ is generally a *linearly accurate* estimate of the eigenvalue λ, i.e.,

$$|r(\mathbf{x}) - \lambda| = \mathcal{O}(\|\mathbf{x} - \mathbf{v}\|_2), \quad \text{as} \quad \mathbf{x} \to \mathbf{v}.$$

(b) If $\mathbf{A}$ is normal, then the Rayleigh quotient $r(\mathbf{x})$ is a *quadratically accurate* estimate of the eigenvalue λ, i.e.,

$$|r(\mathbf{x}) - \lambda| = \mathcal{O}(\|\mathbf{x} - \mathbf{v}\|_2^2), \quad \text{as} \quad \mathbf{x} \to \mathbf{v}.$$

Exercise 3. (Programming, 10 points)

Construct a 4×4 matrix $\mathbf{A}$ by the following Matlab scripts:

```
L=diag([1 2 6 30]); S=randn(4); A=S*L*inv(S);
```

Compare the convergence of Power iteration, Inverse iteration, and Rayleigh quotient iteration. You must use Matlab's `semilogy` to draw three pictures: the x-axis is the iteration index k, and the y-axis is the absolute error of the computed approximate eigenvalues, i.e., $|\lambda^{(k)} - \lambda|$. For each method, stop at the 10th iteration.

Additional Exercise 1. (TreBau Exercise 27.2)

Let $\mathbf{A} \in \mathbb{C}^{m \times m}$ be arbitrary. The set of all Rayleigh quotients of $\mathbf{A}$, corresponding to all nonzero vectors $\mathbf{x} \in \mathbb{C}^m$, is known as the *field of values* or *numerical range* of $\mathbf{A}$, a subset of the complex plane denoted by $\mathcal{W}(\mathbf{A})$. It is well known that $\mathcal{W}(\mathbf{A})$ contains the convex hull of the eigenvalues of $\mathbf{A}$. Prove that if $\mathbf{A}$ is normal, then $\mathcal{W}(\mathbf{A})$ is equal to the convex hull of the eigenvalues of $\mathbf{A}$.