# Numerical Linear Algebra Assignment 8

## Exercise 1. (TreBau Exercise 25.2, 10 points)

Let  $e_1, e_2, e_3, \cdots$  be a sequence of nonnegative numbers representing errors in some iterative process that converge to zero, and suppose there are a constant c and an exponent  $\alpha$  such that for all sufficiently large  $k, e_{k+1} \leq c(e_k)^{\alpha}$ . Then, (1) *linear convergence* or *geometric convergence*:  $\alpha = 1$ and c < 1; (2) *quadratic convergence*:  $\alpha = 2$ ; (3) *cubic convergence*:  $\alpha = 3$ .

- (a) Suppose we want an answer of accuracy  $\mathcal{O}(\varepsilon_{\text{machine}})$ . Assuming the amount of work for each step is  $\mathcal{O}(1)$ , show that the total work requirement in the case of linear convergence is  $\mathcal{O}(|\log(\varepsilon_{\text{machine}})|)$ . How does the constant *c* enter into your work estimate?
- (b) Show that in the case of superlinear convergence, i.e.,  $\alpha > 1$ , the work requirement becomes  $\mathcal{O}(\log(|\log(\varepsilon_{\text{machine}})|))$ . (Hint: The problem may be simplified by defining a new error measure  $f_k = c^{1/(\alpha-1)}e_k$ .) How does the exponent  $\alpha$  enter into your work estimate?

### Exercise 2. (Theorem 1 of Lecture 8, 10 points)

Let  $\{\lambda, \mathbf{v}\}$  be an eigenpair of the matrix  $\mathbf{A} \in \mathbb{C}^{m \times m}$ , i.e.,  $\mathbf{A}\mathbf{v} = \lambda \mathbf{v}$  and  $\mathbf{v} \neq 0$ . We further assume that  $\|\mathbf{v}\|_2 = 1$ . Prove the following:

(a) If **A** is non-normal, then the Rayleigh quotient  $r(\mathbf{x})$  is generally a *linearly accurate* estimate of the eigenvalue  $\lambda$ , i.e.,

$$|r(\mathbf{x}) - \lambda| = \mathcal{O}(||\mathbf{x} - \mathbf{v}||_2), \text{ as } \mathbf{x} \to \mathbf{v}.$$

(b) If **A** is normal, then the Rayleigh quotient  $r(\mathbf{x})$  is a *quadratically accurate* estimate of the eigenvalue  $\lambda$ , i.e.,

 $|r(\mathbf{x}) - \lambda| = \mathcal{O}(||\mathbf{x} - \mathbf{v}||_2^2), \text{ as } \mathbf{x} \to \mathbf{v}.$ 

## Exercise 3. (Programming, 10 points)

Construct a  $4 \times 4$  matrix **A** by the following Matlab scripts:

```
L=diag([1 2 6 30]); S=randn(4); A=S*L*inv(S);
```

Compare the convergence of Power iteration, Inverse iteration, and Rayleigh quotient iteration. You must use Matlab's semilogy to draw three pictures: the x-axis is the iteration index k, and the y-axis is the absolute error of the computed approximate eigenvalues, i.e.,  $|\lambda^{(k)} - \lambda|$ . For each method, stop at the 10th iteration.

#### Additional Exercise 1. (TreBau Exercise 27.2)

Let  $\mathbf{A} \in \mathbb{C}^{m \times m}$  be arbitrary. The set of all Rayleigh quotients of  $\mathbf{A}$ , corresponding to all nonzero vectors  $\mathbf{x} \in \mathbb{C}^m$ , is known as the *field of values* or *numerical range* of  $\mathbf{A}$ , a subset of the complex plane denoted by  $\mathcal{W}(\mathbf{A})$ . It is well known that  $\mathcal{W}(\mathbf{A})$  contains the convex hull of the eigenvalues of  $\mathbf{A}$ . Prove that if  $\mathbf{A}$  is normal, then  $\mathcal{W}(\mathbf{A})$  is equal to the convex hull of the eigenvalues of  $\mathbf{A}$ .