Numerical Linear Algebra Assignment 2

Exercise 1. (10 points)

Show that if $\mathbf{A} \in \mathbb{C}^{m\times n}$, then the maximum singular value of $\mathbf{A}$ satisfies

$$\sigma_{\max}(\mathbf{A}) = \max_{\mathbf{0}\neq\mathbf{x}\in\mathbb{C}^n,\mathbf{0}\neq\mathbf{y}\in\mathbb{C}^m} \frac{|\mathbf{y}^*\mathbf{A}\mathbf{x}|}{\|\mathbf{x}\|_2\|\mathbf{y}\|_2}.$$

(Hint: Cauchy–Schwarz inequality and $\|\mathbf{A}\mathbf{x}\|_2 \le \|\mathbf{A}\|_2\|\mathbf{x}\|_2$.)

Exercise 2. (10 points)

For any $\mathbf{A} \in \mathbb{C}^{m\times l}$ and $\mathbf{B} \in \mathbb{C}^{l\times n}$, show that $\|\mathbf{A}\mathbf{B}\|_{\mathrm{F}} \le \|\mathbf{A}\|_2\|\mathbf{B}\|_{\mathrm{F}}$. (Hint: $\mathbf{A} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^*$, and F-norm is unitarily invariant.)

Exercise 3. (TreBau Exercise 4.5, 10 points)

Show that every $\mathbf{A} \in \mathbb{R}^{m\times n}$ has a real SVD $\mathbf{A} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^\top$ with $\mathbf{U} \in \mathbb{R}^{m\times m}$ and $\mathbf{V} \in \mathbb{R}^{n\times n}$. (Hint: the proof is almost the same as that given in the lecture for the complex case except additional arguments should be given to illustrate all vectors and matrices involved are real.)

Exercise 4. (10 points)

Let $\mathbf{A} \in \mathbb{R}^{m\times n}$. Define

$$\|\mathbf{A}\|_* := \sup_{\mathbf{X}\in\mathbb{R}^{m\times n},\|\mathbf{X}\|_2\le 1} \mathrm{tr}(\mathbf{A}^\top\mathbf{X}).$$

Prove that

$$\|\mathbf{A}\|_* = \sum_{i=1}^r \sigma_i(\mathbf{A}),$$

where $\{\sigma_i(\mathbf{A})\}_{i=1}^r$ are the nonzero singular values of $\mathbf{A}$ and $r = \mathrm{rank}(\mathbf{A})$.

Exercise 5. (10 points)

Let $\sigma_1 \ge \sigma_2 \ge \cdots \ge \sigma_n$ be the singular values of the matrix $\mathbf{A} \in \mathbb{C}^{m\times n}$ $(m \ge n)$. Assume that $z \in \mathbb{C}$. Compute the singular values of the $(m+n)\times(m+n)$ matrix

$$\begin{bmatrix} \mathbf{0} & \mathbf{A} \\ \mathbf{A}^* & z\mathbf{I} \end{bmatrix}.$$

Exercise 6. (10 points)

Let $\mathbf{A} \in \mathbb{C}^{m\times n}$. Show that

$$\left\|\begin{bmatrix} \mathbf{I}_m & \mathbf{A} \\ \mathbf{0} & \mathbf{I}_n \end{bmatrix}\right\|_2 = \sqrt{\frac{2 + \|\mathbf{A}\|_2^2 + \|\mathbf{A}\|_2\sqrt{4 + \|\mathbf{A}\|_2^2}}{2}}.$$

Exercise 7. (10 points)

Show that $\sigma > 0$ is a singular value of $\mathbf{A} \in \mathbb{C}^{m\times n}$ if and only if the block 2×2 matrix

$$\begin{bmatrix} \mathbf{A} & -\sigma\mathbf{I}_m \\ -\sigma\mathbf{I}_n & \mathbf{A}^* \end{bmatrix} \in \mathbb{C}^{(m+n)\times(m+n)}$$

is singular.

Exercise 8. (TreBau Exercise 5.2, 10 points)

Using the SVD, prove that any matrix in $\mathbb{C}^{m\times n}$ is the limit of a sequence of matrices of full rank. In other words, prove that the set of full-rank matrices is a dense subset of $\mathbb{C}^{m\times n}$. Use the 2-norm for your proof. (The norm doesn't matter, since all norms on a finite-dimensional space are equivalent.)

Exercise 9. (10 points)

For a nonsingular matrix $\mathbf{A} \in \mathbb{C}^{n\times n}$, find a singular matrix $\widetilde{\mathbf{A}}$ such that

$$\|\mathbf{A} - \widetilde{\mathbf{A}}\|_2 = \min_{\substack{\mathbf{B}\in\mathbb{C}^{n\times n},\\ \operatorname{rank}(\mathbf{B})<n}} \|\mathbf{A} - \mathbf{B}\|_2.$$

Exercise 10. (10 points)

(i) Prove that any $\mathbf{A} \in \mathbb{C}^{m\times m}$ has a *polar decomposition* $\mathbf{A} = \mathbf{UH}$, where $\mathbf{U} \in \mathbb{C}^{m\times m}$ is a unitary matrix and $\mathbf{H} \in \mathbb{C}^{m\times m}$ is a positive-semidefinite Hermitian matrix.

(ii) Assume that $\mathbf{A}$ is invertible. Prove that $\mathbf{X}_k$ in the following iteration scheme converges to the unitary polar factor $\mathbf{U}$ in the polar decomposition $\mathbf{A} = \mathbf{UH}$. The iteration scheme is:

$$\begin{aligned} \mathbf{X}_0 &= \mathbf{A}, \\ \mathbf{X}_k &= \frac{1}{2}(\mathbf{X}_{k-1} + (\mathbf{X}_{k-1}^{-1})^*), \ k = 1, 2, \ldots \end{aligned}$$

Exercise 11. (TreBau Exercise 11.1, 10 points)

Suppose the $m \times n$ matrix $\mathbf{A}$ has the form

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \end{bmatrix},$$

where $\mathbf{A}_1$ is a nonsingular matrix of dimension $n \times n$ and $\mathbf{A}_2$ is an arbitrary matrix of dimension $(m-n) \times n$. Prove that $\|\mathbf{A}^\dagger\|_2 \le \|\mathbf{A}_1^{-1}\|_2$.

Exercise 12. (10 points)

Let $\mathbf{A} \in \mathbb{C}^{m\times n}$ and $\operatorname{rank}(\mathbf{A}) = r$. Assume that $s \le r$. Solve

$$\min_{\mathbf{Z}\in\mathbb{C}^{n\times m},\ \operatorname{rank}(\mathbf{Z})\le s} \|\mathbf{ZA} - \mathbf{I}_n\|_{\mathrm{F}}^2.$$