Numerical Linear Algebra Assignment 1

Exercise 1. (TreBau Exercise 2.5, 10 points)

Let $\mathbf{S} \in \mathbb{C}^{m\times m}$ be *skew-hermitian*, i.e., $\mathbf{S}^* = -\mathbf{S}$.

(a) Show that the eigenvalues of $\mathbf{S}$ are pure imaginary.

(b) Show that $\mathbf{I} - \mathbf{S}$ is nonsingular.

(c) Show that the matrix $\mathbf{Q} = (\mathbf{I} - \mathbf{S})^{-1}(\mathbf{I} + \mathbf{S})$, know as the *Cayley transform* of $\mathbf{S}$, is unitary. (This is a matrix analogue of a linear fractional transformation $(1 + z)/(1 - z)$, which maps the left half of the complex z-plane conformally onto the unit disk.)

Exercise 2. (TreBau Exercise 2.6, 10 points)

If $\mathbf{u}$ and $\mathbf{v}$ are m-vectors, the matrix $\mathbf{A} = \mathbf{I}+\mathbf{u}\mathbf{v}^*$ is know as a *rank-one perturbation of the identity*. Show that if $\mathbf{A}$ is nonsingular, then its inverse has the form $\mathbf{A}^{-1} = \mathbf{I} + \alpha\mathbf{u}\mathbf{v}^*$ for some scalar α, and give an expression for α. For what $\mathbf{u}$ and $\mathbf{v}$ is $\mathbf{A}$ singular? If it is singular, what is null($\mathbf{A}$)?

Exercise 3. (10 points)

Prove the Cauchy–Schwarz inequality: For any given inner product $\langle\cdot,\cdot\rangle$,

$$|\langle \mathbf{x}, \mathbf{y}\rangle| \le \sqrt{\langle \mathbf{x}, \mathbf{x}\rangle}\sqrt{\langle \mathbf{y}, \mathbf{y}\rangle}.$$

The equality holds if and only if $\mathbf{x}$ and $\mathbf{y}$ are linearly dependent.

Exercise 4. (10 points)

Prove that

$$\langle \mathbf{A}, \mathbf{B}\rangle = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}\overline{b_{ij}}, \quad \forall \mathbf{A}, \mathbf{B} \in \mathbb{C}^{m\times n}$$

is an inner product on $\mathbb{C}^{m\times n}$. (The Frobenius norm on $\mathbb{C}^{m\times n}$ is induced by this inner product.)

Exercise 5. (10 points)

Let $\|\cdot\|$ denote any vector norm on $\mathbb{C}^m$ and $\mathbf{W} \in \mathbb{C}^{m\times m}$ be nonsingular. Prove that $\|\mathbf{x}\|_{\mathbf{W}} = \|\mathbf{W}\mathbf{x}\|$ is a vector norm on $\mathbb{C}^m$.

Exercise 6. (TreBau Exercise 3.2, 10 points)

Let $\|\cdot\|$ denote any vector norm on $\mathbb{C}^m$ and also the induced matrix norm on $\mathbb{C}^{m\times m}$. Show that $\rho(\mathbf{A}) \le \|\mathbf{A}\|$, where $\rho(\mathbf{A})$ is the *spectral radius* of $\mathbf{A}$, i.e., the largest absolute value $|\lambda|$ of an eigenvalue λ of $\mathbf{A}$.

Exercise 7. (TreBau Exercise 3.6, 10 points)

Let $\|\cdot\|$ denote any vector norm on $\mathbb{C}^m$. The corresponding *dual norm* $\|\cdot\|'$ is defined by the formula $\|\cdot\|' = \sup_{\|\mathbf{y}\|=1} |\mathbf{y}^*\mathbf{x}|$.

(a) Prove that $\|\cdot\|'$ is a norm.

(b) Let $\mathbf{x}, \mathbf{y} \in \mathbb{C}^m$ with $\|\mathbf{x}\| = \|\mathbf{y}\| = 1$ be given. Show that there exists a rank-one matrix $\mathbf{B} = \mathbf{y}\mathbf{z}^*$ such that $\mathbf{B}\mathbf{x} = \mathbf{y}$ and $\|\mathbf{B}\| = 1$, where $\|\mathbf{B}\|$ is the matrix norm of $\mathbf{B}$ induced by the vector norm $\|\cdot\|$. You may use the following lemma, without proof: given $\mathbf{x} \in \mathbb{C}^m$, there exists a nonzero $\mathbf{z} \in \mathbb{C}^m$ such that $|\mathbf{z}^*\mathbf{x}| = \|\mathbf{z}\|'\|\mathbf{x}\|$.

Exercise 8. (10 points)

Let $\mathbf{A} \in \mathbb{C}^{m\times n}$, $\mathbf{B} \in \mathbb{C}^{n\times r}$ and let $\|\cdot\|_\alpha$, $\|\cdot\|_\beta$, and $\|\cdot\|_\gamma$ be norms on $\mathbb{C}^m, \mathbb{C}^n$, and $\mathbb{C}^r$, respectively. Prove the induced matrix norms $\|\cdot\|_{\alpha,\gamma}$, $\|\cdot\|_{\alpha,\beta}$, and $\|\cdot\|_{\beta,\gamma}$ satisfy $\|\mathbf{AB}\|_{\alpha,\gamma} \leq \|\mathbf{A}\|_{\alpha,\beta}\|\mathbf{B}\|_{\beta,\gamma}$.

Exercise 9. (10 points)

Prove that $\|\mathbf{A}\|_{\infty,1} = \max_{i,j} |a_{ij}|$.

Exercise 10. (TreBau Exercise 3.4, 10 points)

Let $\mathbf{A}$ be an $m \times n$ matrix and let $\mathbf{B}$ be a submatrix of $\mathbf{A}$, that is, an $s \times t$ matrix ($s \leq m, t \leq n$) obtained by selecting certain rows and columns of $\mathbf{A}$.

(a) Explain how $\mathbf{B}$ can be obtained by multiplying $\mathbf{A}$ by certain row and column "deletion matices" as in step 7 of Exercise 1.1.

(b) Using this product, show that $\|\mathbf{B}\|_p \leq \|\mathbf{A}\|_p$ for any p with $1 \leq p \leq \infty$.